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A Bilinear Vector Integral. I*

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The subject of spectral theory requires a wide variety of integrals with tremendous disparities in their properties. The only properties in common to the various integrals $\int_M f(m)\, E(dm)$ are the linearity in the function f being integrated, the linearity in the additive set function E defined on a field Σ of subsets of some set M and some kind of continuity of the integrand $fE(\sigma)$ varying with each case. In this paper we shall define a bilinear vector integral that includes all of the integrals, such as the Hilbert transform, the Calderón–Zygmund integrals, improper integrals in general, and many more. An example will be given to illustrate some surprising and unexpected properties of the integral.

The basic notation and assumptions used throughout the paper are as follows. All scalars are complex numbers, i.e., belong to one-dimensional unitary space E^1. Let $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{Z}$ be three complex Banach spaces and let $z = \langle y, x\rangle$ be a bilinear form defined for $x \in \mathfrak{X}$, $y \in \mathfrak{Y}$, and $z \in \mathfrak{Z}$. It is assumed that this form is continuous in the sense that

$$|\langle y, x\rangle| \leqslant L\,|\,x\,|\,|\,y\,|, \qquad x \in \mathfrak{X},\ y \in \mathfrak{Y}, \tag{1}$$

for some $L \geqslant 0$. M is an arbitrary set without topology, Σ is a field of subsets of the set M, E is a bounded finitely additive $\mathfrak{X}$-valued set function defined on Σ. In defining the integral we purposely use the somewhat bizarre notation

$$\int_M \langle f(m), E(dm)\rangle, \tag{2}$$

since this will stress the consequences of the bilinearity of the integrand and perhaps dissuade the reader from erroneous conjectures suggested by the more conventional symbol $\int_M f(m)\, E(dm)$. A function f on M

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to $\mathfrak{Y}$ is said to be Σ-measurable if the inverse image $f^{-1}(G) \in \Sigma$ for every open set G in $\mathfrak{Y}$. A function $f\colon M \to \mathfrak{Y}$ is called a *Σ-measurable step function* if it has the form

$$f(m) = \sum_{j=1}^{n} y_j \chi_{\sigma_j}(m), \qquad m \in M, \tag{3}$$

where $\sigma_1, \ldots, \sigma_n$ are mutually disjoint sets in M; χ_{σ_j} is the characteristic function of σ_j; and $y_1, \ldots, y_n$ are vectors in $\mathfrak{Y}$. We define the integral of the function (3) by

$$\int_M \langle f(m), E(dm) \rangle = \sum_{j=1}^{n} \langle y_j, E(\sigma_j) \rangle. \tag{4}$$

The right-hand side of Eq. (4) is easily shown to be independent of the particular form in which the function (4) is written. For a function f on M to $\mathfrak{Y}$ having the form

$$f(m) = \lim_{n\to\infty} f_n(m) \qquad \text{uniformly on } M, \tag{5}$$

where f_n, $n = 1, \ldots$, are Σ-measurable step functions, we would like to define the integral of f by

$$\int_M \langle f(m), E(dm) \rangle = \lim_{n\to\infty} \int_M \langle f_n(m), E(dm) \rangle. \tag{6}$$

Unfortunately, this limit rarely exists. This phenomenon is of common occurrence in the theory of self adjoint operators whose spectrums are discrete sets of eigenvalues having no finite point of accumulation. The following example is quite elementary and will clearly illustrate our point.

Illustrative Example. Let M be the set of all positive integers, let Σ be the σ-field of all subsets of M, let $\mathfrak{X} = l_2$, and let E be the countably additive self adjoint spectral measure in l_2 defined for $\sigma \in \Sigma$ and $x = \{\xi_n\} \in l_2$ by the equations $E(\sigma) = 0$ if σ is void, and for a nonvoid σ, $E(\sigma)x$ is that vector in l_2 whose components coincide with those of x for $n \in \sigma$ and whose components are zero for $n \notin \sigma$. Let $\mathfrak{X} = \mathfrak{Y} = l_2$, let $\mathfrak{Z}$ be the complex number system E^1, and let $\langle y, x \rangle = (y, x)$, $x, y \in l_2$. Let $E(m)$ be the projection $E(\sigma)$ if σ consists of the single point m. Let $y_n = x_n$ be the point in l_2 whose nth component is 1 and whose other components are zero. Then the sequence $\{f_n\}$ of Σ-measurable step functions defined by the equations

$$f_n(m) = \epsilon_n \frac{y_m}{(m)^{1/2}}, \qquad 1 \leqslant m \leqslant n,$$

$$f_n(m) = 0, \qquad n < m,$$

$$\epsilon_n = \left(\sum_{j=1}^{n} \frac{1}{j} \right)^{-1/4},$$

has the property that

$$\sup_m |f_n(m)|^2 = \left(\sum_{j=1}^{n} \frac{1}{j} \right)^{-1/2} \to 0.$$

But the integrals

$$\int_M |f_n(m)|^2 (E(dm)\, x_j, x_j) = \epsilon_n^2 \sum_{m=1}^{n} \left| \frac{y_m}{(m)^{1/2}} \right|^2 (E(m)\, x_j, x_j)$$

$$= \left(\sum_{j=1}^{n} \frac{1}{j} \right)^{1/2} \to \infty.$$

This example shows that even when $\mathfrak{X} = \mathfrak{Y} = l_2$, which has the nicest properties of all infinite dimensional B-spaces, and when $\mathfrak{Z} = E^1$, we can have (6) violated in the sense that

$$\sup_m |f_n(m)| \to 0, \left| \int_M \langle f_n(m), E(dm) \rangle \right| \to \infty.$$

For a function h with values in a B-space we use the symbol $|h|_\infty$ for the supremum of the norms of all vectors in the range of h. Thus, for functions $f: M \to \mathfrak{Y}$ and $E_0: \Sigma_0 \to \mathfrak{X}$ the norms

$$|f|_\infty = \sup_{m \in M} |f(m)|, \qquad |E_0|_\infty = \sup_{\sigma \in \Sigma_0} |E_0(\sigma)| \tag{7}$$

are finite if and only if f and E_0 are bounded.

Definition 1. A $\mathfrak{Y}$-valued function f on M is said to be *integrable* with respect to the $\mathfrak{X}$-valued bounded additive set function E_0 defined on the field Σ_0 of subsets of M if

(i) f is Σ_0-measurable,

(ii) the closure of the range of f is compact in $\mathfrak{Y}$, and

(iii) there is a constant K such that for every $\mathfrak{Y}$-valued Σ_0-measurable step function f on M

(iv) $|\int_M \langle f(m), E_0(dm)\rangle| \leqslant K |f|_\infty |E_0|_\infty$.

The class of such integrable functions f will be denoted by the symbol $И(M, \Sigma_0, E_0, \mathfrak{Y})$.

The Russian letter И is used, since it is the first letter in the Russian words for integrable, integral, and integration.

One of the salient features of the definition of an integrable function lies in its explicit focusing of attention on the inequality (iv), whose verification is one of the major problems in integration theory. We know of no verification of (iv) that covers a large variety of the spaces involved.

The definition of the integral of an integrable function is as follows.

THEOREM 2. *For every f in $И(M, \Sigma_0, E_0, \mathfrak{Y})$ there is a sequence $\{f_n\}$ of $\mathfrak{Y}$-valued Σ_0-measurable step functions converging to f uniformly on M for which the limit*

$$\lim_n \int_M \langle f_n(m), E_0(dm)\rangle$$

exists. Furthermore, this limit is the same for any sequence of Σ_0-measurable step functions that converges to f uniformly on M. The integral of f is unambiguously defined by the equation

$$\int_M \langle f(m), E_0(dm)\rangle = \lim_{n\to\infty} \int_M \langle f_n(m), E_0(dm)\rangle,$$

and is a linear map of $И(M, \Sigma_0, E_0, \mathfrak{Y})$ into $\mathfrak{Z}$ satisfying the inequality

$$\left|\int_M \langle f(m), E_0(dm)\rangle\right| \leqslant K |f|_\infty |E_0|_\infty . \tag{8}$$

The proof of the theorem is quite elementary.

If the spaces $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{Z}$ are B-algebras, commutative B-algebras, B^*-algebras, etc., and if E_0 satisfies certain conditions, one can state many additional properties of the integral. The general change of measure principle is quite useful and its proof is quite elementary.

THEOREM 3. *Let A and B be two bounded additive $\mathfrak{X}$-valued set functions defined on Σ and related by the identity*

(i) $A(\delta) = B(h^{-1}(\delta)), \ \delta \in \Sigma,$

where h is a mapping of M into itself satisfying the identity

(ii) $h^{-1}(\delta) \in \Sigma, \ \delta \in \Sigma.$

Then for every f in $\mathfrak{U}(M, \Sigma, A, \mathfrak{Y})$, the function $f(h(.))$ is in $\mathfrak{U}(M, \Sigma, B, \mathfrak{Y})$ and

(iii) $\int_M \langle f(m), A(dm)\rangle = \int_M \langle f(h(m)), B(dm)\rangle.$

In the general context of Definition 1 of the integral, one can make what corresponds to the Borel extension of an $\mathfrak{X}$-valued or extended real valued additive set function on the field Σ_0 of sets in M. In addition, one can make what corresponds to the Lebesgue extension of the Borel extension to a set function E_1 on a field Σ_1 containing Σ_0 and the Lebesgue extension of the pair E_1, Σ_1 is E_1, Σ_1.

There are many bilinear integrals not found among the familiar integrals one meets in classical analysis. Perhaps the best known and most powerful of these are the principal value convolutions of Calderón and Zygmund. They have many deep applications. But they are not bilinear vector integrals of the type we are using. However, many mathematicians have defined bilinear vector integrals $\int_S f(s)\, g(ds)$ where the function f and the measure g are defined on some type of measure space and have their values in Banach spaces. The following list of references is quite incomplete [1–7]. A more complete list will be found when the details of this paper are published. For those readers interested in a much more complete list and the properties of and relations between the various integrals, the reader is referred to [1], which gives just such a discussion.

The problems that have led to the integral presented here are, literally, dozens of nonself-adjoint problems arising in the theory of neutron diffusion with or without boundary conditions as well as other similar diffusion problems, not one of which, with a few exceptions, can be solved with the theory of spectral operators simply because the operators involved are not spectral operators. The excellent treatise of Morse and Feshback [7] gives many examples arising in diffusion theory of the type we are studying but not one of them, to the author's knowledge, has been given a satisfactory mathematical explanation. This is not surprising since at the time this treatise was written the only known spectral theory was that for self adjoint operators. The space $\mathfrak{X}$ in these problems is a topological linear vector space which is not metrizable. Later, when the details of this paper are published, the integral will be defined for certain general categories of nonmetrizable topological linear vector spaces. The particular spaces used in diffusion theory will not be specified until they are needed in a third paper treating this subject.

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